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journal homepage: www.elsevier.com/locate/discEnumeration of $(0, 1)$ -matrices avoiding some 2×2 matricesHyeong-Kwan Ju^a, Seunghyun Seo^{b,*}^a Department of Mathematics, Chonnam National University, Kwangju 500-757, South Korea^b Department of Mathematics Education, Kangwon National University, Chuncheon 200-701, South Korea

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ABSTRACT

We enumerate the number of $(0, 1)$ -matrices avoiding 2×2 submatrices satisfying certain conditions. We also provide corresponding exponential generating functions.

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1. Introduction

Let $M(k, n)$ be the set of $k \times n$ matrices with entries 0 and 1. It is obvious that the number of elements in the set $M(k, n)$ is 2^{kn} . It would be interesting to consider the number of elements in $M(k, n)$ with certain conditions. For example, how many matrices of $M(k, n)$ do not have 2×2 submatrices of the forms $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$? In this article we will give answers to the previous question and other questions.

Consider $M(2, 2)$, the set of all possible 2×2 submatrices. For two elements P and Q in $M(2, 2)$, we denote $P \sim Q$ if Q can be obtained from P by row or column exchanges. It is obvious that \sim is an equivalence relation on $M(2, 2)$. With this equivalence relation, $M(2, 2)$ is partitioned with seven equivalent classes having the following seven representatives.

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \Gamma = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \quad L = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \quad J = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Here C , T , and L mean “corner”, “top”, and “left”, respectively. Let \mathcal{S} be the set of these representatives, i.e.,

$$\mathcal{S} := \{I, \Gamma, C, T, L, J, O\}.$$

Given an element S in \mathcal{S} , a matrix A is an element of the set $M(S)$ if and only if for every permutation π_1 of the rows and π_2 of the columns, the resulting matrix does not have the submatrix S . Equivalently, $A \in M(S)$ means that A has no submatrices in the equivalent class $[S]$. For a subset α of \mathcal{S} , $M(\alpha)$ is defined by the set $\bigcap_{S \in \alpha} M(S)$. Note that the definition of $M(S)$ (also $M(\alpha)$) is different from that in [13,17]. If A belongs to $M(\alpha)$, then we say that A avoids α . We let $\phi(k, n; \alpha)$ be the number of $k \times n(0, 1)$ -matrices in $M(\alpha)$.

Our goal is to express $\phi(k, n; \alpha)$ in terms of k and n explicitly for each subset α of the set \mathcal{S} . For $|\alpha| = 1$, we can easily notice that $\phi(k, n; \Gamma) = \phi(k, n; C)$ and $\phi(k, n; J) = \phi(k, n; O)$ by swapping 0 and 1. We also notice that $\phi(k, n; T) = \phi(n, k; L)$

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by transposing the matrices. The number $\phi(k, n; I)$ is well known (see [2,7,8]) and $(0, 1)$ -matrices avoiding type I are called $(0, 1)$ -lonesum matrices (we will define and discuss this in 2.2). In fact, lonesum matrices are the primary motivation of this article and its corresponding work. The study of $M(J)$ (equivalently $M(O)$) appeared in [9,13,17], but finding a closed form of $\phi(k, n; J) = \phi(k, n; O)$ is still open. The notion of “ Γ -free matrix” was introduced by Spinrad [16]. He dealt with a totally balanced matrix which has a permutation of the rows and columns that are Γ -free. We remark that the set of totally balanced matrices is different from $M(\Gamma)$.

In this paper we calculate $\phi(k, n; \alpha)$, where α 's are $\{\Gamma\}$ (equivalently $\{C\}$) and $\{T\}$ (equivalently $\{L\}$). We also enumerate $M(\alpha)$ where α 's are $\{\Gamma, C\}$, $\{T, L\}$, and $\{J, O\}$. For the other subsets of \mathcal{A} , we discuss them briefly in the last section. Note that some of our result (Section 3.5) is an independent derivation of some of the results in [11, Section 3] by Kitaev et al.; for other relevant papers see [10,12].

2. Preliminaries

2.1. Definitions and notations

A matrix P is called $(0, 1)$ -matrix if all the entries of P are 0 or 1. From now on we will consider $(0, 1)$ -matrices only, so we will omit “ $(0, 1)$ ” if it causes no confusion. Let $M(k, n)$ be the set of $k \times n$ -matrices. Clearly, if $k, n \geq 1$, $M(k, n)$ has 2^{kn} elements. For convention we assume that $M(0, 0) = \{\emptyset\}$ and $M(k, 0) = M(0, n) = \emptyset$ for positive integers k and n .

Given a matrix P , a submatrix of P is formed by selecting certain rows and columns from P . For example, if $P = \begin{pmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \end{pmatrix}$, then $P(2, 3; 2, 4) = \begin{pmatrix} f & h \\ j & l \end{pmatrix}$.

Given two matrices P and Q , we say P contains Q , whenever Q is equal to a submatrix of P . Otherwise say P avoids Q . For example, $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ contains $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ but avoids $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. For a matrix P and a set α of matrices, we say that P avoids the type set α if P avoids all the matrices in α . If it causes no confusion we will simply say that P avoids α .

Given a set α of matrices, let $\phi(k, n; \alpha)$ be the number of $k \times n$ matrices avoiding α . From the definition of $M(k, n)$, for any set α , we have $\phi(0, 0; \alpha) = 1$ and $\phi(k, 0; \alpha) = \phi(0, n; \alpha) = 0$ for positive integers k and n . Let $\Phi(x, y; \alpha)$ be the bivariate exponential generating function for $\phi(k, n; \alpha)$, i.e.,

$$\Phi(x, y; \alpha) := \sum_{n \geq 0} \sum_{k \geq 0} \phi(k, n; \alpha) \frac{x^k y^n}{k! n!} = 1 + \sum_{n \geq 1} \sum_{k \geq 1} \phi(k, n; \alpha) \frac{x^k y^n}{k! n!}.$$

Let $\Phi(z; \alpha)$ be the exponential generating function for $\phi(n, n; \alpha)$, i.e.,

$$\Phi(z; \alpha) := \sum_{n \geq 0} \phi(n, n; \alpha) \frac{z^n}{n!}.$$

Given $f, g \in \mathbb{C}[[x, y]]$, we denote $f \stackrel{2}{=} g$ if the coefficients of $x^k y^n$ in f and g are the same, for each $k, n \geq 2$.

2.2. I -avoiding matrices (Lonesum matrices)

This is related to the lonesum matrices. A lonesum matrix is a $(0, 1)$ -matrix determined uniquely by its column-sum and row-sum vectors. For example, $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$ is a lonesum matrix since it is a unique matrix determined by the column-sum vector $(2, 0, 3)$ and the row-sum vector $(2, 1, 2)^t$. However $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ is not, since $\begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ has the same column-sum vector $(2, 0, 2)$ and row-sum vector $(2, 1, 1)^t$.

Theorem 2.1 (Brewbaker [2]). *A matrix is a lonesum matrix if and only if it avoids I .*

Theorem 2.1 implies that $\phi(k, n; I)$ is equal to the number of $k \times n$ lonesum matrices.

Definition 2.2. Bernoulli number B_n is defined as following:

$$\sum_{n \geq 0} B_n \frac{x^n}{n!} = \frac{x e^x}{e^x - 1}.$$

Note that B_n can be written explicitly as

$$B_n = \sum_{m=0}^n (-1)^{m+n} \frac{m! S(n, m)}{m+1},$$

[1]	[0]	[0]	[0*]
[0]	[1]	[0]	[0*]
[0]	[0]	[1]	[0*]
[0*]	[0*]	[0*]	[0*]

Fig. 1. A matrix avoiding Γ can be changed into a block diagonal matrix.

where $S(n, m)$ is the Stirling number of the second kind. The poly-Bernoulli number, introduced first by Kaneko [7], is defined as

$$\sum_{n \geq 0} B_n^{(k)} \frac{x^n}{n!} = \frac{\text{Li}_k(1 - e^{-x})}{1 - e^{-x}},$$

where the polylogarithm $\text{Li}_k(x)$ is the function $\text{Li}_k(x) := \sum_{m \geq 1} \frac{x^m}{m^k}$. Bernoulli numbers are nothing but poly-Bernoulli numbers with $k = 1$. Sanchez-Peregrino [15] proved that $B_n^{(-k)}$ has the following simple expression:

$$B_n^{(-k)} = \sum_{m=0}^{\min(k,n)} (m!)^2 S(n+1, m+1) S(k+1, m+1).$$

Brewbaker [2] and Kim et al. [8] proved that the number of $k \times n$ lonesum matrices is the poly-Bernoulli number $B_n^{(-k)}$, which yields the following result.

Proposition 2.3 (Brewbaker [2]; Kim et al. [8]). *The number of $k \times n$ matrices avoiding I is equal to $B_n^{(-k)}$, i.e.,*

$$\phi(k, n; I) = \sum_{m=0}^{\min(k,n)} (m!)^2 S(n+1, m+1) S(k+1, m+1). \quad (1)$$

The generating function $\Phi(x, y; I)$, given by Kaneko [7], is

$$\Phi(x, y; I) = e^{x+y} \sum_{m \geq 0} [(e^x - 1)(e^y - 1)]^m = \frac{e^{x+y}}{e^x + e^y - e^{x+y}}. \quad (2)$$

Also, $\Phi(z; I)$ can be easily obtained as follows:

$$\begin{aligned} \Phi(z; I) &= \sum_{n \geq 0} \phi(n, n; I) \frac{z^n}{n!} = \sum_{n \geq 0} \sum_{m \geq 0} (-1)^{m+n} m! S(n, m) (m+1)^n \frac{z^n}{n!} \\ &= \sum_{m \geq 0} (-1)^m m! \sum_{n \geq 0} S(n, m) \frac{(-(m+1)z)^n}{n!} \\ &= \sum_{m \geq 0} (1 - e^{-(m+1)z})^m. \end{aligned} \quad (3)$$

3. Main results

3.1. Γ -avoiding matrices (or C)

By row exchange and column exchange we can change the original matrix into a block matrix as in Fig. 1. Here **[0]** (resp. **[1]**) stands for a 0-block (resp. 1-block) and **[0*]** stands for a 0-block or an empty block. Diagonal blocks are **[1]**'s except for the last block **[0*]**, and the off-diagonal blocks are **[0]**'s.

Theorem 3.1. *The number of $k \times n$ matrices avoiding Γ is given by*

$$\phi(k, n; \Gamma) = \sum_{m=0}^{\min(k,n)} m! S(n+1, m+1) S(k+1, m+1). \quad (4)$$

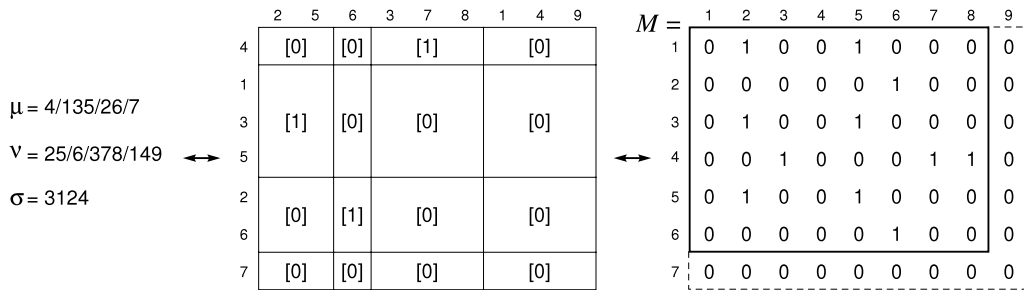


Fig. 2. A matrix avoiding Γ corresponds to two set partitions with a permutation.

Proof. Let $\mu = \{C_1, C_2, \dots, C_{m+1}\}$ be a set partition of $[n+1]$ into $m+1$ blocks. Here the block C_i 's are ordered by the largest element of each block. Thus $n+1$ is contained in C_{m+1} . Likewise, let $\nu = \{D_1, D_2, \dots, D_{m+1}\}$ be a set partition of $[k+1]$ into $m+1$ blocks. Choose $\sigma \in S_{m+1}$ with $\sigma(m+1) = m+1$, where S_{m+1} is the set of all permutations of length $m+1$. Given (μ, ν, σ) we define a $k \times n$ matrix $M = (a_{i,j})$ by

$$a_{i,j} := \begin{cases} 1, & (i,j) \in C_l \times D_{\sigma(l)} \text{ for some } l \in [m] \\ 0, & \text{otherwise} \end{cases}.$$

It is obvious that the matrix M avoids the type Γ .

Conversely, let M be a $k \times n$ matrix avoiding type Γ . Set $(k+1) \times (n+1)$ matrix \tilde{M} by augmenting zeros to the last row and column of M . By row exchange and column exchange we can change \tilde{M} into a block diagonal matrix B , where each diagonal is 1-block except for the last diagonal. By tracing the position of columns (resp. rows) in \tilde{M} , B gives a set partition of $[n+1]$ (resp. $[k+1]$). Let $\{C_1, C_2, \dots, C_{m+1}\}$ (resp. $\{D_1, D_2, \dots, D_{m+1}\}$) be the set partition of $[n+1]$ (resp. $[k+1]$). Note that the block C_i 's and D_i 's are ordered by the largest element of each block. Let σ be a permutation on $[m]$ defined by $\sigma(i) = j$ if C_i and D_j form a 1-block in B .

The number of set partitions π of $[n+1]$ is $S(n+1, m+1)$, and the number of set partitions π' of $[k+1]$ is $S(n+1, k+1)$. The cardinality of the set of σ 's is the cardinality of S_m , i.e., $m!$. Since the number of blocks $m+1$ runs through 1 to $\min(k, n) + 1$, the sum of $S(k+1, m+1)S(n+1, m+1)m!$ gives the required formula. \square

Example 1. Let $\mu = 4/135/26/7$ be a set partition of $[7]$ and $\nu = 25/6/378/149$ of $[9]$ into 4 blocks. Let $\sigma = 3124$ be a permutation in S_4 such that $\sigma(4) = 4$. From (μ, ν, σ) we can construct the 6×8 matrix M which avoids type Γ as in Fig. 2.

To find the generating function for $\phi(k, n; \Gamma)$ the following formula (see [5]) is helpful.

$$\sum_{n \geq 0} S(n+1, m+1) \frac{x^n}{n!} = e^x \frac{(e^x - 1)^m}{m!}. \quad (5)$$

From Theorem 3.1 and (5), we can express $\Phi(x, y; \Gamma)$ as follows:

$$\begin{aligned} \Phi(x, y; \Gamma) &= \sum_{n, k \geq 0} \sum_{m \geq 0} m! S(n+1, m+1) S(k+1, m+1) \frac{x^k}{k!} \frac{y^n}{n!} \\ &= \sum_{m \geq 0} \frac{1}{m!} \sum_{k \geq 0} m! S(k+1, m+1) \frac{x^k}{k!} \sum_{n \geq 0} m! S(n+1, m+1) \frac{y^n}{n!} \\ &= \sum_{m \geq 0} \frac{1}{m!} e^x (e^x - 1)^m e^y (e^y - 1)^m \\ &= \exp[(e^x - 1)(e^y - 1) + x + y]. \end{aligned} \quad (6)$$

Remark 1. It seems to be difficult to find a simple expression of $\Phi(z; \Gamma)$. The sequence $\phi(n, n; \Gamma)$ is not listed in the OEIS [14]. The first few terms of $\phi(n, n; \Gamma)$ ($0 \leq n \leq 9$) are as follows:

1, 2, 12, 128, 2100, 48032, 1444212, 54763088, 2540607060, 140893490432, ...

3.2. $\{\Gamma, C\}$ -avoiding matrices

Given the equivalence relation \sim on $M(2, 2)$, which is defined in Section 1, if we define the new equivalent relation $P \sim' Q$ by $P \sim Q$ or $P \sim \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} - Q$, then $[\Gamma] \cup [C]$ becomes a single equivalent class. Clearly $\phi(k, n; \{\Gamma, C\})$ is the number

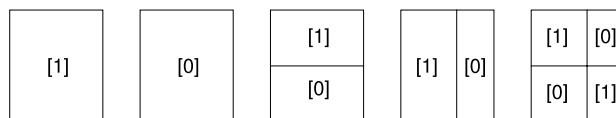


Fig. 3. Possible reduced forms of matrices avoiding $\{\Gamma, C\}$.

of $k \times n$ (0, 1)-matrix which does not have a submatrix in $[\Gamma] \cup [C]$. From now on we simply write $\phi(k, n; \Gamma, C)$, instead of $\phi(k, n; \{\Gamma, C\})$. The reduced form of a matrix M avoiding $\{\Gamma, C\}$ is very simple as in Fig. 3. In this case if the first row and the first column of M are determined then the rest of the entries of M are determined uniquely. Hence the number $\phi(k, n; \Gamma, C)$ of such matrices is

$$\phi(k, n; \Gamma, C) = 2^{k+n-1} \quad (k, n \geq 1), \quad (7)$$

and its exponential generating function is

$$\Phi(x, y; \Gamma, C) = 1 + \frac{1}{2} (e^{2x} - 1)(e^{2y} - 1). \quad (8)$$

Clearly, $\phi(n, n; \Gamma, C) = 2^{2n-1}$ for $n \geq 1$. Thus its exponential generating function is

$$\Phi(z; \Gamma, C) = \frac{1}{2} (e^{4z} + 1). \quad (9)$$

3.3. T -avoiding matrices (or L)

Given a (0, 1)-matrix, 1-column (resp. 0-column) is a column in which all entries consist of 1's (resp. 0's). We denote a 1-column (resp. 0-column) by $\mathbf{1}$ (resp. $\mathbf{0}$). A mixed column is a column which is neither $\mathbf{0}$ nor $\mathbf{1}$. For $k = 0$, we have $\phi(0, n; T) = \delta_{0,n}$. In case $k \geq 1$, i.e., there being at least one row, we can enumerate as follows:

- case 1: there are no mixed columns. Then each column should be $\mathbf{0}$ or $\mathbf{1}$. The number of such $k \times n$ matrices is 2^n .
- case 2: there is one mixed column. In this case each column should be $\mathbf{0}$ or $\mathbf{1}$ except for one mixed column. The number of $k \times n$ matrices of this case is $2^{n-1} n (2^k - 2)$.
- case 3: there are two mixed columns. As in case 2, each column should be $\mathbf{0}$ or $\mathbf{1}$ except for two mixed columns, say, v_1 and v_2 . The number of $k \times n$ matrices of this case is the sum of the following three subcases:
 - $v_1 + v_2 = \mathbf{1}$: $2^{n-2} \binom{n}{2} 2! S(k, 2)$
 - $v_1 + v_2$ has an entry 0: $2^{n-2} \binom{n}{2} 3! S(k, 3)$
 - $v_1 + v_2$ has an entry 2: $2^{n-2} \binom{n}{2} 3! S(k, 3)$
- case 4: there are m ($m \geq 3$) mixed columns v_1, \dots, v_m . The number of $k \times n$ matrices of this case is the sum of the following four subcases:
 - $v_1 + \dots + v_m = \mathbf{1}$: $2^{n-m} \binom{n}{m} m! S(k, m)$
 - $v_1 + \dots + v_m = (m-1)\mathbf{1}$: $2^{n-m} \binom{n}{m} m! S(k, m)$
 - $v_1 + \dots + v_m$ has an entry 0: $2^{n-m} \binom{n}{m} (m+1)! S(k, m+1)$
 - $v_1 + \dots + v_m$ has an entry m : $2^{n-m} \binom{n}{m} (m+1)! S(k, m+1)$

Adding up all numbers in the previous cases yields the following theorem.

Theorem 3.2. For $k, n \geq 1$ the number of $k \times n$ matrices avoiding T is given by

$$\phi(k, n; T) = 2 \sum_{l \geq 1} \binom{n}{l-1} l^k + (n^2 - n - 4) 2^{n-2} - n(n+3) 2^{n+k-3}. \quad (10)$$

Proof.

$$\begin{aligned} \phi(k, n; T) &= 2^n + 2^{n-1} \binom{n}{1} (2^k - 2) + 2^{n-2} \binom{n}{2} (2! S(k, 2) + 3! 2 S(k, 3)) \\ &\quad + \sum_{m=3}^n 2^{n-m+1} \binom{n}{m} (m! S(k, m) + (m+1)! S(k, m+1)) \\ &= 2 \sum_{m=0}^n 2^{n-m} \binom{n}{m} m! S(k+1, m+1) + (n^2 - n - 4) 2^{n-2} - n(n+3) 2^{n+k-3} \\ &= 2 \sum_{l \geq 1} \binom{n}{l-1} l^k + (n^2 - n - 4) 2^{n-2} - n(n+3) 2^{n+k-3}. \quad \square \end{aligned}$$

Note that in the proof of [Theorem 3.2](#) we use the identity

$$\sum_{m \geq 0} \binom{n}{m} m! S(k, m) 2^{n-m} = \sum_{l \geq 0} \binom{n}{l} l^k,$$

where both sides count the number of functions f from $[k]$ to $[n]$ such that each element of $[n] \setminus f([k])$ has two colors.

The generating function $\Phi(x, y; T)$ is given by

$$\begin{aligned} \Phi(x, y; T) &= 1 + \sum_{n \geq 1} \sum_{k \geq 1} 2 \sum_{l \geq 1} \binom{n}{l-1} l^k \frac{x^k y^n}{k! n!} \\ &\quad + \sum_{n \geq 1} \sum_{k \geq 1} (n^2 - n - 4) 2^{n-2} \frac{x^k y^n}{k! n!} - \sum_{n \geq 1} \sum_{k \geq 1} n(n+3) 2^{n+k-3} \frac{x^k y^n}{k! n!} \\ &= 1 + \left(2e^x (e^{y(e^x+1)} - 1) - 2e^{2y} + 2 \right) + (e^x - 1) ((y^2 - 1)e^{2y} + 1) - \frac{1}{2} y(y+2)e^{2y}(e^{2x} - 1) \\ &= 2e^{y(e^x+1)+x} - \frac{y^2 + 2y}{2} e^{2x+2y} + (y^2 - 1)e^{x+2y} - e^x - \frac{y^2 - 2y + 2}{2} e^{2y} + 2. \end{aligned} \quad (11)$$

Note that if we use the symbol “ $\stackrel{2}{=}$ ” introduced in [Section 2.1](#), then

$$\Phi(x, y; T) \stackrel{2}{=} 2e^{y(e^x+1)+x} - \frac{y^2 + 2y}{2} e^{2x+2y} + (y^2 - 1)e^{x+2y}.$$

For the $n \times n$ matrices we have

$$\phi(n, n; T) = 2 \sum_{l \geq 1} \binom{n}{l-1} l^n + (n^2 - n - 4) 2^{n-2} - n(n+3) 2^{2n-3}.$$

Thus the generating function $\Phi(n, n; T)$ is given by

$$\begin{aligned} \sum_{n \geq 0} \phi(n, n; T) \frac{z^n}{n!} &= 2 \sum_{n \geq 0} \sum_{l \geq 1} \binom{n+1}{l} l^{n+1} \frac{z^n}{(n+1)!} + \sum_{n \geq 0} \frac{n^2 - n - 4}{4} \frac{(2z)^n}{n!} - \frac{n(n+3)}{8} \frac{(4z)^n}{n!} \\ &= \frac{2}{z} \sum_{l \geq 1} \frac{(lz)^l}{l!} \sum_{n \geq l-1} \frac{(lz)^{n-l+1}}{(n-l+1)!} + (z^2 - 1)e^{2z} - 2z(z+1)e^{4z} \\ &= \frac{2}{z} \sum_{l \geq 1} \frac{l^l}{l!} (ze^z)^l + (z^2 - 1)e^{2z} - 2z(z+1)e^{4z} \\ &= \frac{2}{z} (ze^z W'(-ze^z)) + (z^2 - 1)e^{2z} - 2z(z+1)e^{4z} \\ &= \frac{-2W(-ze^z)}{z + zW(-ze^z)} + (z^2 - 1)e^{2z} - 2z(z+1)e^{4z}, \end{aligned} \quad (12)$$

where

$$W(x) := \sum_{n \geq 1} (-n)^{n-1} \frac{x^n}{n!}$$

is the Lambert W -function which is the inverse function of $f(W) = We^W$. See [\[3\]](#) for an extensive study about the Lambert W -function.

Remark 2. The sequence $\phi(n, n; T)$ is not listed in the OEIS [\[14\]](#). The first few terms of $\phi(n, n; T)$ ($0 \leq n \leq 9$) are as follows:

1, 2, 14, 200, 3536, 67472, 1423168, 34048352, 927156224, 28490354432, ...

3.4. $\{T, L\}$ -avoiding matrices

Given the equivalence relation \sim on $M(2, 2)$, which is defined in [Section 1](#), if we define the new equivalent relation $P \sim'' Q$ by $P \sim Q$ or $P \sim Q^t$, then $[T] \cup [L]$ becomes a single equivalent class. Clearly $\phi(k, n; \{T, L\})$ is the number of $k \times n(0, 1)$ -matrix which does not have any submatrix in $[T] \cup [L]$. By the symmetry of $\{T, L\}$, we have

$$\phi(k, n; T, L) = \phi(n, k; T, L).$$

So it is enough to consider the case $k \geq n$. For $k \leq 2$ or $n \leq 1$, we have

$$\begin{aligned}\phi(0, n; T, L) &= \delta_{0,n}, & \phi(1, n; T, L) &= 2^n, \\ \phi(k, 0; T, L) &= \delta_{k,0}, & \phi(k, 1; T, L) &= 2^k, \\ \phi(2, 2; T, L) &= 12.\end{aligned}$$

Given a $(0, 1)$ -vector v with a length of at least 3, v is called 1-dominant (resp. 0-dominant) if all entries of v are 1's (resp. 0's) except one entry.

Theorem 3.3. For $k \geq 3$ and $n \geq 2$, the number of $k \times n$ matrices avoiding $\{T, L\}$ is equal to twice the number of rook positions in the $k \times n$ chessboard. In other words,

$$\phi(k, n; T, L) = 2 \sum_{m=0}^{\min(k,n)} \binom{k}{m} \binom{n}{m} m!. \quad (13)$$

Proof. Suppose M is a $k \times n$ matrix avoiding $\{T, L\}$. It is easy to show each of the following steps:

- (i) If M has a mixed column v , then v should be either 0-dominant or 1-dominant.
- (ii) Assume that v is 0-dominant. This implies that other mixed columns (if any) in M should be 0-dominant.
- (iii) Any non-mixed column in M should be a 0-column.
- (iv) The location of 1's in M corresponds to a rook position in the $k \times n$ chessboard.

If we assume v is 1-dominant in (ii) then the locations of 0's again corresponds to a rook position. The summand of RHS in (13) is the number of rook positions in the $k \times n$ chessboard with m rooks. This completes the proof. \square

The generating function $\Phi(x, y; T, L)$ is given by

$$\Phi(x, y; T, L) = 2e^{xy+x+y} - \frac{(xy)^2}{2} - 2xy + 3 - 2e^x - 2e^y + x(e^y - 2y - 1)(e^y - 1) + y(e^x - 2x - 1)(e^x - 1). \quad (14)$$

Here the crucial part of the Eq. (14) can be obtained as follows:

$$\begin{aligned}\sum_{k,n \geq 0} \left(\sum_{m \geq 0} \binom{k}{m} \binom{n}{m} m! \right) \frac{x^k y^n}{k! n!} &= \sum_{m \geq 0} m! \left(\sum_{k \geq 0} \binom{k}{m} \frac{x^k}{k!} \right) \left(\sum_{n \geq 0} \binom{n}{m} \frac{y^n}{n!} \right) \\ &= \sum_{m \geq 0} m! \left(\frac{x^m}{m!} e^x \right) \left(\frac{y^m}{m!} e^y \right) = \exp(xy + x + y).\end{aligned}$$

Note that if we use the symbol “ $\stackrel{2}{=}$ ” introduced in Section 2.1, then

$$\Phi(x, y; T, L) \stackrel{2}{=} 2e^{xy+x+y} - \frac{(xy)^2}{2}.$$

For the $n \times n$ matrices we have

$$\begin{aligned}\phi(0, 0; T, L) &= 1, & \phi(1, 1; T, L) &= 2, & \phi(2, 2; T, L) &= 12, \quad \text{and} \\ \phi(n, n; T, L) &= 2 \sum_{m=0}^n \binom{n}{m}^2 m!. \quad (n \geq 3)\end{aligned}$$

Thus the generating function $\Phi(z; T, L)$ is given by

$$\Phi(z; T, L) = \frac{2e^{\frac{z}{1-z}}}{1-z} - 1 - 2z - z^2. \quad (15)$$

Note that we use the equation

$$\sum_{n \geq 0} \left(\sum_{m=0}^n \binom{n}{m}^2 m! \right) \frac{z^n}{n!} = \frac{e^{\frac{z}{1-z}}}{1-z},$$

which appears in [4, pp. 597–598].

Table 1
The sequence $\phi(k, n; J, O)$.

k	n							
	1	2	3	4	5	6	7	...
1	2	4	8	16	32	64	128	...
2	4	14	44	128	352	928	2,368	...
3	8	44	156	408	720	720	0	0
4	16	128	408	840	720	720	0	0
5	32	352	720	720	0	0	0	0
6	64	928	720	720	0	0	0	0
7	128	2,368	0	0	0	0	0	0
⋮	⋮	⋮	0	0	0	0	0	0

3.5. $\{J, O\}$ -avoiding matrices

Recall the equivalence relation \sim' defined in Section 3.2. With this relation, $\{J, O\}$ becomes a single equivalent class. Due to the symmetry of $\{J, O\}$ it is obvious that

$$\phi(k, n; J, O) = \phi(n, k; J, O).$$

The k -color bipartite Ramsey number $br(G; k)$ of a bipartite graph G is the minimum integer n such that in any k -coloring of the edges of $K_{n,n}$ there is a monochromatic subgraph isomorphic to G . Beineke and Schwenk [1] had shown that $br(K_{2,2}; 2) = 5$. From this we can see that

$$\phi(k, n; J, O) = 0 \quad (k, n \geq 5).$$

For $k = 1$ and 2, we have

$$\phi(1, n; J, O) = 2^n, \quad \phi(2, n; J, O) = (n^2 + 3n + 4)2^{n-2}.$$

Note that the sequence $(n^2 + 3n + 4)2^{n-2}$ appears in [14, A007466] and its exponential generating function is $(1+x)^2 e^{2x}$. For $k \geq 3$, we have

$$\begin{aligned} \phi(3, n; J, O) &= \phi(4, n; J, O) = 0 \quad \text{for } n \geq 7, \\ \phi(5, n; J, O) &= \phi(6, n; J, O) = 0 \quad \text{for } n \geq 5, \\ \phi(k, n; J, O) &= 0 \quad \text{for } k \geq 7 \text{ and } n \geq 3. \end{aligned}$$

For exceptional cases, due to the symmetry of $\{J, O\}$, it is enough to consider the following:

$$\begin{aligned} \phi(3, 3; J, O) &= 156, & \phi(3, 4; J, O) &= 408, & \phi(4, 4; J, O) &= 840, \\ \phi(3, 5; J, O) &= \phi(3, 6; J, O) = \phi(4, 5; J, O) = \phi(4, 6; J, O) &= 720. \end{aligned}$$

The sequence $\phi(k, n; J, O)$ is listed in Table 1. Note that Kitaev et al. have already calculated $\phi(k, n; J, O)$ in [11, Proposition 5], but the numbers of $\phi(3, 3; J, O)$ and $\phi(4, 4; J, O)$ are different from ours.

The generating function $\Phi(x, y; J, O)$ is given by

$$\begin{aligned} \Phi(x, y; J, O) &= 1 + x e^{2y} + y e^{2x} + x^2(1+y)^2 e^{2y} + y^2(1+x)^2 e^{2x} \\ &\quad - \left(x + y + \frac{x^2}{2!} + \frac{y^2}{2!} + 2xy + 2x^2y + 2xy^2 + 14 \frac{x^2y^2}{2!2!} \right) + 156 \frac{x^3y^3}{3!3!} + 840 \frac{x^4y^4}{4!4!} \\ &\quad + 720 \left(\frac{x^3y^5}{3!5!} + \frac{x^5y^3}{5!3!} + \frac{x^3y^6}{3!6!} + \frac{x^6y^3}{6!3!} + \frac{x^4y^5}{4!5!} + \frac{x^5y^4}{5!4!} + \frac{x^4y^6}{4!6!} + \frac{x^6y^4}{6!4!} \right). \end{aligned} \quad (16)$$

In particular, the generating function $\Phi(z; J, O)$ is given by

$$\Phi(z; J, O) = 1 + 2z + 7z^2 + 26z^3 + 35z^4. \quad (17)$$

4. Concluding remarks

Table 2 summarizes our results (except the I -avoiding case). Due to the amount of difficulty, we are not able to enumerate the number $\phi(k, n; J)$, hence $\phi(k, n; O)$. Note that $\phi(k, n; J)$ is equal to the following:

- The number of labeled (k, n) -bipartite graphs with girth of at least 6, i.e., the number of C_4 -free labeled (k, n) -bipartite graphs, where C_4 is a cycle of length 4.
- The cardinality of the set $\{(B_1, B_2, \dots, B_k) : B_i \subseteq [n] \forall i, |B_i \cap B_j| \leq 1 \forall i \neq j\}$.

Table 2
Formulas and generating functions avoiding α .

α	$\phi(k, n; \alpha)$	$\Phi(x, y; \alpha)$	$\Phi(z; \alpha)$
I	(1)	(2)	(3)
Γ (or C)	(4)	(6)	complicated
$\{\Gamma, C\}$	(7)	(8)	(9)
T (or L)	(10)	(11)	(12)
$\{T, L\}$	(13)	(14)	(15)
J (or O)	unknown	unknown	unknown
$\{J, O\}$	Table 1	(16)	(17)

For the other subsets α of \mathcal{S} which is not listed in Table 2, we have calculated $\phi(k, n; \alpha)$ in [6]. Note that if the size of α increases then enumeration of $M(\alpha)$ becomes easier.

For further research, we suggest the following problems.

1. In addition to $(0, 1)$ -matrices, one can consider $(0, 1, \dots, r)$ -matrices with $r \geq 2$.
2. Consideration of the results of adding the line sum condition to each individual case given in the first column of Table 2.

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References

- [1] L.W. Beineke, A.J. Schwenk, On a bipartite form of the Ramsey problem, Proceedings of the 5th British Combinatorial Conference, 1975, Congr. Numer. XV (1975).
- [2] C. Brewbaker, A combinatorial interpretation of the poly-Bernoulli numbers and two Fermat analogues, *Integers* 8 (2008) A02.
- [3] R.M. Corless, G.H. Gonnet, D.E.G. Hare, D.J. Jeffrey, D.E. Knuth, On the Lambert function, *Advances in Computational Mathematics* 5 (1996) 329–359.
- [4] P. Flajolet, R. Sedgewick, *Analytic Combinatorics*, Cambridge University Press, 2009.
- [5] R. Graham, D. Knuth, O. Patashnik, *Concrete Mathematics*, second ed., Addison-Wesley, 1994.
- [6] H.-K. Ju, S. Seo, Counting $(0, 1)$ -matrices avoiding α of size greater than 1, preprint, available at: <http://shseo.info/>.
- [7] M. Kaneko, Poly-Bernoulli numbers, *Journal de Théorie des Nombres de Bordeaux* 9 (1997) 221–228.
- [8] H.-K. Kim, D. Krotov, J.-Y. Lee, Poly-Bernoulli numbers and lonesum matrices, available at: [arXiv:1103.4884v1](https://arxiv.org/abs/1103.4884v1).
- [9] J.-R. Kim, U. Peled, I. Perepelitsa, V. Pless, S. Friedland, Explicit construction of families of LDPC codes with no 4-cycles, *IEEE Transactions on Information Theory* 50 (10) (2004) 2378–2388.
- [10] S. Kitaev, On multi-avoidance of right angled numbered polyomino patterns, *Integers* 4 (2004) A21.
- [11] S. Kitaev, T. Mansour, A. Vella, Pattern Avoidance in Matrices, *Journal of Integer Sequences* 8 (2) (2005) Article 05.2.2.
- [12] S. Kitaev, J.R. Robbins, On multi-dimensional patterns, *Pure Mathematics and Applications. PU.M.A.* 18 (3–4) (2007) 291–299.
- [13] B. Klinz, R. Rudolf, G.J. Woeginger, Permuting matrices to avoid forbidden submatrices, *Discrete Applied Mathematics* 60 (1995) 223–248.
- [14] Online Encyclopedia of Integer Sequence, available at: <http://oeis.org/>.
- [15] R. Sanchez-Peregrino, Closed formula for poly-Bernoulli numbers, *Fibonacci Quarterly* 40 (4) (2002) 362–364.
- [16] J.P. Spinrad, Nonredundant 1's in Γ -free matrices, *SIAM Journal on Discrete Mathematics* 8 (2) (1995) 251–257.
- [17] J.P. Spinrad, Efficient Graph Representations, in: *Fields Institute Monographs*, vol. 19, American Mathematical Society, 2003.